

# Existence of the $D0-D4$ Bound State: a detailed Proof\*

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## Abstract

We consider the supersymmetric quantum mechanical system which is obtained by dimensionally reducing  $d=6$ ,  $N=1$  supersymmetric gauge theory with gauge group  $U(1)$  and a single charged hypermultiplet. Using the deformation method and ideas introduced by Poratti and Rozenberg [1], we present a detailed proof of the existence of a normalizable ground state for this system.

## 1 Introduction

The particular system, which we will consider, belongs to a class of supersymmetric quantum mechanical models. These models appear in the study of quantized membranes [2],  $D$ -brane bound states [3], and M-theory [4]. Especially the question of existence respectively absence of normalizable ground states, i.e., zero energy states, is of physical importance. The Hamiltonian of these models is of the form

$$H = -\Delta + V + H_F .$$

The scalar potential  $V$  is polynomial in the bosonic degrees of freedom and admits zero energy valleys extending to infinity while  $H_F$  is quadratic in the fermionic degrees of freedom and linear in the bosonic degrees of freedom. Moreover, the Hilbert space carries a unitary representation of a gauge group. The physical Hilbert space consists of gauge

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invariant states. Due to supersymmetric cancellations, the zero energy valleys render the Hamiltonian to have continuous spectrum, which covers the positive real axis. Therefore, the Hamiltonian is non-Fredholm and the question about existence of ground states is subtle. The Witten index  $I_W$ , i.e., the number of bosonic ground states minus the number of fermionic ground states, can be calculated by means of

$$I_W = \lim_{R \rightarrow \infty} \lim_{\beta \rightarrow \infty} \text{Tr}((-1)^F \chi_R e^{-\beta H}) ,$$

where  $\chi_R$  denotes the characteristic function of the ball of radius  $R$  centered around the origin in configuration space, c.p. [5]. Since there is no gap in the spectrum one has to deal with a delicate analysis of boundary contributions. As a different approach, Poratti and Rozenberg proposed in [1] a deformation method to detect the existence of normalizable ground states for systems with at least two real supercharges. One deforms the supercharges of the system with a real potential  $w$ ,

$$D \rightarrow D_w := e^{-w} D e^w , \quad D^\dagger \rightarrow D_w^\dagger := e^w D^\dagger e^{-w} ,$$

such that the spectrum of the deformed Hamiltonian  $H_w := D_w D_w^\dagger + D_w^\dagger D_w$  becomes discrete. This might allow one to show the existence of a ground state  $\Psi_w$  for the deformed problem. Provided that  $e^{\pm w} \Psi_w$  is normalizable, then, the original problem admits a ground state as well. Using this method the number of ground states for numerous models could be determined, [6].

In this paper, we consider the quantum mechanical system which is obtained by dimensionally reducing  $N = 1$  supersymmetric gauge theory, with gauge group  $U(1)$  and with a single charged hypermultiplet from six dimensions. The system appears in the problem of counting H-monopole ground states in the toroidally compactified heterotic string [7]. Moreover, the same system describes the low energy dynamics of a  $D0$ -brane in the presence of a  $D4$ -brane [8, 9]. String duality arguments predict the existence of exactly one bound state at threshold for this system, c.p. [10]. The existence of such a state provides a check of the correctness of these duality hypotheses.

In [5], an analysis was sketched of how to obtain the value one for the Witten index for this system. Combined with vanishing Theorems, [11], such a result implies that the model has a unique ground state. Independently of the work in [5], it was argued in [1] how a deformation method may be used to establish existence of a ground state. In this paper we use this deformation method and follow the main ideas of [1] to present a rigorous proof of the existence of a ground state. In particular, we make the argument in [1] mathematically precise in two important aspects. First we prove the existence of a ground state for the deformed problem: we have to do semiclassical analysis on the space of gauge invariant functions and we have to deal with the fact that  $H_F$  is unbounded. In a second part we prove a decay estimate for the ground state of the deformed problem. In particular, we show that it decays sufficiently fast implying that the original problem also has a ground state. To obtain this decay property, we use an Agmon [12] estimate and combine it with a symmetry argument. We think this is a clear and direct way to obtain the necessary decay. Alternatively one could also determine the asymptotic form of the ground state by analysing the effective dynamics along a potential valley. Such an analysis was indicated in [1]. Similarly one could use a supercharge analysis related

to the one in [13] (which was used to determine the asymptotic form of the bound state of two  $D0$ -branes). Similar considerations have to be taken into account when using the deformation method to study the number of zero energy states for other supersymmetric models of the same type. Moreover, there are results about the structure of the  $D0$ - $D4$  bound state [14].

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we introduce the deformation method and give an outline of the proof, which is then presented in Section 4.

## 2 The model

The model is obtained by dimensionally reducing  $N = 1$ ,  $U(1)$  supersymmetric gauge theory with a single charged hypermultiplet, from  $5 + 1$  dimensions to  $0 + 1$  dimension [8, 14]. The bosonic degrees of freedom are given by

$$q = (q_j)_{j=1,\dots,4} \in \mathbb{R}^4, \quad \text{and} \quad x = (x^\mu)_{\mu=1,\dots,5} \in \mathbb{R}^5,$$

and their configuration space is  $X = \mathbb{R}^4 \times \mathbb{R}^5$ . Let  $p_j$ ,  $j = 1, \dots, 4$ , and  $p^\mu$ ,  $\mu = 1, \dots, 5$ , be the associated canonical momenta obeying,

$$[q_j, p_k] = i\delta_{jk}, \quad [x^\mu, p^\nu] = i\delta^{\mu\nu}.$$

The fermionic degrees of freedom are described by the real Clifford generators

$$\lambda_a, a = 1, \dots, 8 \quad \text{and} \quad \psi_a, a = 1, \dots, 8,$$

i.e.,  $\lambda_a^\dagger = \lambda_a$ ,  $\psi_a^\dagger = \psi_a$ , and

$$\{\lambda_a, \lambda_b\} = \delta_{ab}, \quad \{\psi_a, \psi_b\} = \delta_{ab}, \quad \{\lambda_a, \psi_b\} = 0.$$

(Here and below  $\{\cdot, \cdot\}$  stands for the anticommutator.) By  $\mathcal{F}$  we denote the irreducible representation space of this Clifford algebra. The dimension of  $\mathcal{F}$  is  $2^8$ . We introduce as a preliminary Hilbert space

$$\mathcal{H}_0 = L^2(X; \mathcal{F}) = L^2(X) \otimes \mathcal{F}.$$

As given in Appendix A, we choose an explicit real irreducible representation

$$\gamma^\mu = (\gamma_{ab}^\mu)_{a,b=1,\dots,8}, \quad \mu = 1, \dots, 5,$$

of the gamma matrices in 5 dimensions, i.e.,

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}.$$

Furthermore we consider the real  $8 \times 8$  matrices

$$s^i = (s_{ab}^i)_{a,b=1,\dots,8}, \quad i = 1, \dots, 4,$$

as they are defined in Appendix A. We note that  $s^1 = \mathbb{1}_{8 \times 8}$  and  $(s^l)^T = -s^l$  for  $l = 2, 3, 4$  and that each  $s^i$  commutes with the  $\gamma$ -matrices. We define

$$D_{ab} = \frac{1}{2}(q^R s^2 \bar{q}^R)_{ab},$$

with

$$\begin{aligned} q^R &= s^1 q_1 + s^2 q_2 + s^3 q_3 + s^4 q_4, \\ \bar{q}^R &= s^1 \bar{q}_1 - s^2 \bar{q}_2 - s^3 \bar{q}_3 - s^4 \bar{q}_4. \end{aligned}$$

We will use the convention of summing over repeated indices. The supercharges are given by

$$Q_a = (s^j \psi)_a p_j + (\gamma^\mu \lambda)_a p^\mu + D_{ab} \lambda_b + (\gamma^\mu s^j s^2 \psi)_a x^\mu q_j, \quad a = 1, \dots, 8.$$

Note, for any  $8 \times 8$  matrix  $A$  we set  $(A\psi)_a = A_{ab}\psi_b$ ,  $\psi A\psi = \psi_a A_{ab}\psi_b$ , and likewise for expressions containing  $\lambda_a$ . The Hilbert space  $\mathcal{H}_0$  carries a unitary representation of  $U(1)$ , called the gauge transformation, defined by the generator

$$J = W_{12} + W_{34} - \frac{i}{2} \psi s^2 \bar{\psi},$$

where  $W_{ij} = q_i p_j - q_j p_i$ . We set  $|x| := (x^\mu x^\mu)^{1/2}$  and  $|q| := (q_i q_i)^{1/2}$ . The full model satisfies

$$\{Q_a, Q_b\} = \delta_{ab} H + 2\gamma^\mu_{ab} x^\mu J, \quad (1)$$

with

$$\begin{aligned} H &= p^\mu p^\mu + p_i p_i + |x|^2 |q|^2 + \frac{1}{4} |q|^4 - i x^\mu \psi \gamma^\mu s^2 \psi + i 2 q_j \lambda s^j s^2 \psi \\ &= -\Delta + V + H_F, \end{aligned}$$

where we have defined

$$V = |x|^2 |q|^2 + \frac{1}{4} |q|^4, \text{ and } H_F = -i x^\mu \psi \gamma^\mu s^2 \psi + i 2 q_j \lambda s^j s^2 \psi.$$

The Hilbert space of the model  $\mathcal{H}$  is the  $U(1)$ -invariant subspace of  $\mathcal{H}_0$ , i.e.,

$$\mathcal{H} = \{\Psi \in \mathcal{H}_0 \mid J\Psi = 0\}.$$

Note that the supercharges  $Q_a$  are  $U(1)$  invariant and that on  $\mathcal{H}$  the superalgebra (1) closes, i.e.,

$$\{Q_a, Q_b\}|_{\mathcal{H}} = \delta_{ab} H|_{\mathcal{H}}.$$

The Hilbert space  $\mathcal{H}_0$  carries a natural representation of  $Spin(5)$  defined by the infinitesimal generators

$$T^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - \frac{i}{4} \gamma_{ab}^{\mu\nu} (\lambda_a \lambda_b + \psi_a \psi_b), \quad \mu, \nu = 1, \dots, 5,$$

with  $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ . Under this representation the supercharges  $Q_a$  transform as spinors and the Hamiltonian  $H$  is invariant. The action of  $Spin(5)$  commutes with the gauge transformation, and thus leaves the Hilbert space  $\mathcal{H}$  invariant.

We introduce the fermionic number operator  $(-1)^F := 2^8 \lambda_1 \lambda_2 \dots \lambda_8 \psi_1 \psi_2 \dots \psi_8$ , which anti-commutes with  $Q_a$  and commutes with  $H$ , and decompose the Hilbert space by means of  $(-1)^F$  as

$$\mathcal{H}_\pm := \{\Psi \in \mathcal{H} \mid (-1)^F \Psi = \pm \Psi\},$$

i.e., into bosonic (+) and fermionic (-) sectors.

We note that the operators  $Q_a$  and  $H$  are essentially self adjoint on  $C_0^\infty(X; \mathcal{F})$ . Furthermore their restriction to  $\mathcal{H}$  is essentially self adjoint on the space of  $U(1)$ -invariant functions in  $C_0^\infty(X; \mathcal{F})$ .

### 3 Result and outline of the proof

The main Theorem is the following:

**Theorem 1.** *There exists a state  $\Psi \in \mathcal{H}$  with  $H\Psi = 0$ .*

To prove this theorem, we use the deformation method introduced in [1]. We consider the “complex” supercharges  $D$  and  $D^\dagger$ ,

$$D = \frac{1}{\sqrt{2}}(Q_1 + iQ_2), \quad D^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2).$$

On  $\mathcal{H}$ ,  $D^2 = D^{\dagger 2} = 0$  and

$$H = \{D, D^\dagger\}. \tag{2}$$

We define the  $U(1)$ -invariant function  $w_k$  on  $X$ , by

$$w_k = k \cdot x^1, \quad \text{for } k \geq 0.$$

We introduce the deformed supercharges

$$D_k = e^{-w_k} D e^{w_k}, \quad D_k^\dagger = e^{w_k} D^\dagger e^{-w_k}.$$

We have

$$D_k = D - k \frac{1}{\sqrt{2}}((\gamma^1 \lambda)_1 + i(\gamma^1 \lambda)_2), \quad D_k^\dagger = D^\dagger + k \frac{1}{\sqrt{2}}((\gamma^1 \lambda)_1 - i(\gamma^1 \lambda)_2).$$

As a little calculation shows, we have on  $\mathcal{H}$

$$H_k = \{D_k, D_k^\dagger\}, \quad \text{with } H_k := H + k^2 + k(q_3^2 + q_4^2) - k(q_1^2 + q_2^2).$$

We point out that the deformed Hamiltonian is  $Spin(5)$  invariant, despite that the function  $w_k = k \cdot x^1$  is not.

The claim of Theorem 1 is an immediate consequence of the following three propositions.

**Proposition 2.** *If for some  $k$  there exists a state  $\Psi_k \in \mathcal{H}$  with  $H_k \Psi_k = 0$  such that  $e^{\pm w_k} \Psi_k \in \mathcal{H}$ , then  $H \Psi = 0$  for some state  $\Psi \in \mathcal{H}$ .*

**Remark.** Proposition 2 holds for more general supersymmetric quantum mechanical systems and deformations, c.p. [1].

The proof of Proposition 2, which is presented in Subsection 4.1, makes use of the Hodge decomposition and a cohomology argument.

**Proposition 3.** *For  $k$  large enough, there exists a unique state  $\Psi \in \mathcal{H}$  with  $H_k \Psi = 0$ .*

**Remark.** Proposition 3 implies that  $H_k$  admits a zero energy ground state for all  $k > 0$ . This follows from the stability of the Fredholm index of the continuous family of Fredholm operators,  $(0, \infty) \ni k \mapsto A_k := 2^{-1/2}(D_k + D_k^\dagger)|_{\mathcal{H}_-} : \mathcal{H}_- \rightarrow \mathcal{H}_+$ , where the topology is given by the graph norm with respect to  $A_0$ , see for example [15]. However, we will not use this fact to prove Theorem 1.

To prove Proposition 3, which is done in Subsection 4.2, we first observe that the set of points, in which the scalar potential of the deformed Hamiltonian, i.e.,

$$V_k = V + k^2 - k(q_1^2 + q_2^2) + k(q_3^2 + q_4^2),$$

vanishes, is a circle in configuration space  $X$  (see e.g. (5)). Its radius is proportional to  $k^{1/2}$ . The circle is an orbit of the  $U(1)$  action on  $X$ . In the direction orthogonal to the circle the Hessian of  $V_k$  is non degenerate. Note that up to gauge transformations the scalar potential vanishes exactly in one point. Moreover, at infinity the potential  $V_k$  is bounded below by  $k^2$ . Using semiclassical analysis of eigenvalues, as given for example in [16], together with a gauge fixing procedure, we show that there exists only one low lying eigenvalue for  $k \rightarrow \infty$ . In particular, we have to consider the fact that  $H_F$  is unbounded from below. By supersymmetry this low lying eigenvalue must equal zero for large  $k$ .

**Proposition 4.** *For  $k > 0$ , a state  $\Psi \in \mathcal{H}$  with  $H_k \Psi = 0$  satisfies  $e^{\pm w_k} \Psi \in \mathcal{H}$ .*

For the proof of Proposition 4, which is given in Subsection 4.3, we need to show that  $\Psi$  decays sufficiently fast as  $|x| \rightarrow \infty$ . We write the Hamiltonian as a sum of a free Laplacian in the  $x$ -variables and an  $x$ -dependent operator, which describes the dynamics in the transverse direction. We show that the latter is bounded below by  $k^2 - c|x|^{-2}$  for some constant  $c$  and  $|x|$  large. Using an Agmon estimate we then conclude that

$$|x|^{-1} e^{k|x|} \Psi$$

is square integrable at infinity. As will be shown, this together with the fact that  $\Psi$  is invariant under  $Spin(5)$  yields  $e^{\pm w_k} \Psi \in \mathcal{H}$ .

**Remark.** To be precise, the operators  $D, D^\dagger, D_k, D_k^\dagger$  and  $H_k$  are defined in  $\mathcal{H}_0$  and  $\mathcal{H}$  as the closure on  $C_0^\infty(X; \mathcal{F})$  and  $C_0^\infty(X; \mathcal{F}) \cap \mathcal{H}$ , respectively. The domain of  $D$  is the set of all  $\Psi$  in  $\mathcal{H}_0$  and  $\mathcal{H}$  such that  $D\Psi$  (defined in the sense of distributions) is again in  $\mathcal{H}_0$  and  $\mathcal{H}$ , respectively (and analogous for the domains of  $D^\dagger, D_k, D_k^\dagger, H$ , and  $H_k$ ). Indeed,  $D^\dagger$  (resp.  $D_k^\dagger$ ) is the adjoint of  $D$  (resp.  $D_k$ ).

## 4 Proofs

### 4.1 Proof of Proposition 2

We shall first show the Hodge decomposition

$$\mathcal{H} = \ker H \oplus \overline{\text{Ran } D} \oplus \overline{\text{Ran } D^\dagger}. \quad (3)$$

To show the orthogonality, we note that

$$(D\Psi, D^\dagger\Phi) = (D^2\Psi, \Phi) = 0,$$

with  $\Psi \in \mathcal{D}(D)$  and  $\Phi \in \mathcal{D}(D^\dagger)$ , and  $\Psi \in \ker H$  iff  $D\Psi = 0$  and  $D^\dagger\Psi = 0$ , by (2). To show the completeness, we note that for each  $\Psi \in (\ker H)^\perp$ ,

$$\begin{aligned} \Psi &= \lim_{a \downarrow 0} P_{(a, \infty)}(H)\Psi \\ &= \lim_{a \downarrow 0} \frac{1}{2}(DD^\dagger + D^\dagger D)\frac{1}{H}P_{(a, \infty)}(H)\Psi \\ &= \lim_{a \downarrow 0} \left( \frac{1}{2}D(D^\dagger \frac{1}{H}P_{(a, \infty)}(H)\Psi) + \frac{1}{2}D^\dagger(D \frac{1}{H}P_{(a, \infty)}(H)\Psi) \right) \\ &= \lim_{a \downarrow 0} \frac{1}{2}D(D^\dagger \frac{1}{H}P_{(a, \infty)}(H)\Psi) + \lim_{a \downarrow 0} \frac{1}{2}D^\dagger(D \frac{1}{H}P_{(a, \infty)}(H)\Psi). \end{aligned}$$

By  $P_\Omega(H)$  we denoted the projection valued measure of  $H$ , and the last equality follows since the two terms belong to different orthogonal subspaces. Hence we have shown (3).

The equation  $H_k\Psi_k = 0$  implies  $D_k\Psi_k = 0$  and  $D_k^\dagger\Psi_k = 0$ , and further  $De^{w_k}\Psi_k = 0$  and  $D^\dagger e^{-w_k}\Psi_k = 0$ . Assume  $\ker H = \{0\}$ . Then

$$e^{w_k}\Psi_k \in \ker D = \text{Ran } D^\dagger^\perp = \overline{\text{Ran } D}$$

by the Hodge decomposition. It follows that  $e^{w_k}\Psi_k = \lim_{n \rightarrow \infty} D\Phi_n$  for some  $\Phi_n$ , but then

$$(\Psi_k, \Psi_k) = (e^{w_k}\Psi_k, e^{-w_k}\Psi_k) = \lim_{n \rightarrow \infty} (D\Phi_n, e^{-w_k}\Psi_k) = \lim_{n \rightarrow \infty} (\Phi_n, D^\dagger e^{-w_k}\Psi_k) = 0.$$

This is a contradiction, and hence  $\ker H \neq \{0\}$ . □

### 4.2 Proof of Proposition 3

We shall first rescale the operators  $H_k$ ,  $D_k$  and  $D_k^\dagger$ . For  $\Psi \in \mathcal{H}$  and  $t > 0$ , we define the unitary operator

$$(U(t)\Psi)(\xi) = t^{9/2}\Psi(t\xi),$$

where  $\xi = (q, x)$ . Furthermore, we define

$$\begin{aligned} K_t &:= t^{2/3}U(t^{1/3})H_{t^{2/3}}U^*(t^{1/3}) \\ F_t &:= t^{1/3}U(t^{1/3})D_{t^{2/3}}U^*(t^{1/3}) \\ F_t^\dagger &:= t^{1/3}U(t^{1/3})D_{t^{2/3}}^\dagger U^*(t^{1/3}). \end{aligned}$$

It follows that on  $\mathcal{H}$

$$\{F_t, F_t^\dagger\} = K_t \quad , \quad F_t^2 = 0 \quad , \quad F_t^{\dagger 2} = 0 \quad ,$$

and

$$K_t = -\Delta + t^2 V_1 + t H_F \quad ,$$

where

$$V_1 = |x|^2 |q|^2 + \frac{1}{4} |q|^4 + 1 + (q_3^2 + q_4^2) - (q_1^2 + q_2^2) \quad .$$

Proposition 3 follows from

**Lemma 5.** *Let  $E_n(t)$  denote the  $n$ th eigenvalue of  $K_t$  counting multiplicity. Then*

$$\lim_{t \rightarrow \infty} E_1(t)/t = 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} E_2(t)/t \geq r > 0 \quad . \quad (4)$$

By supersymmetry, each non-zero eigenvalue of  $K_t$  must be two fold degenerate, i.e., occur as the eigenvalue of a pair consisting of a bosonic and a fermionic eigenvector (see Theorem 6.3., [16]). In view of (4), for large  $t$ , two fold degeneracy of  $E_1(t)$  is not possible. Hence  $E_1(t) = 0$ . Moreover, this eigenvalue is nondegenerate.

*Proof of Lemma 5.* Writing the deformed potential  $V_1$  as

$$V_1 = |x|^2 |q|^2 + \left( \frac{1}{2} (q_1^2 + q_2^2) - 1 \right)^2 + (q_3^2 + q_4^2) \left( 1 + \frac{1}{4} (q_3^2 + q_4^2) + \frac{1}{2} (q_1^2 + q_2^2) \right) \quad (5)$$

we see that the set of points  $\Gamma$  in which the potential  $V_1$  vanishes is given by

$$\begin{aligned} \Gamma &:= \{(q, x) \in X \mid V_1(q, x) = 0\} \\ &= \{(q, x) \in X \mid q_1^2 + q_2^2 = 2, q_3 = 0, q_4 = 0, x = 0\} \quad . \end{aligned}$$

The set  $\Gamma$  is a circle in the  $(q_1, q_2)$ -plane about the origin with radius  $\sqrt{2}$ . The Hessian of  $V_1$  at points lying in  $\Gamma$  is

$$(\text{Hess } V_1)_{\alpha\beta}|_\Gamma = \left. \left( \frac{\partial^2 V_1}{\partial \xi^\alpha \partial \xi^\beta} \right) \right|_\Gamma = \begin{pmatrix} (2q_r q_s) & 0 & 0 \\ 0 & 4\mathbb{I}_{2 \times 2} & 0 \\ 0 & 0 & 4\mathbb{I}_{5 \times 5} \end{pmatrix} \quad , \quad \alpha, \beta = 1, \dots, 9 \quad ,$$

with  $(\xi^1, \dots, \xi^9) := (q_1, \dots, q_4, x^1, \dots, x^5)$  and  $r, s = 1, 2$ . At a point  $p \in \Gamma$ , the tangent to  $\Gamma$  is the only degenerate direction of the Hessian.

To show that there exists only one low lying eigenvalue, we will fix the  $U(1)$  gauge. For  $\omega \in L^1(X)$  with  $(W_{12} + W_{34})\omega = 0$ , we may integrate out the coordinate  $q_1$  as follows. We introduce the coordinates

$$\begin{aligned} \Phi : [0, 2\pi] \times [0, \infty) \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^4 \\ \begin{pmatrix} \alpha \\ \rho \\ v_3 \\ v_4 \end{pmatrix} &\longmapsto \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 \\ \rho \\ v_3 \\ v_4 \end{pmatrix} \quad , \end{aligned}$$

with  $\alpha = \arctan(q_2/q_1)$  and  $\rho = (q_1^2 + q_2^2)^{1/2}$ . The metric determinant is  $\sqrt{\det D\Phi^T D\Phi} = |\det D\Phi| = \rho$ , and

$$\int_{\mathbb{R}^4 \times \mathbb{R}^5} dq_1 \dots dq_4 d^5x \omega(q, x) = 2\pi \int_{(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^5} d\rho dv_3 dv_4 d^5x \rho \omega((0, \rho, v_3, v_4), x) .$$

The integration on the right hand side is reduced to the gauge fixed configuration space  $\widehat{X} := \{0\} \times (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^5 \subset X$ . We introduce the Hilbert space

$$\widehat{\mathcal{H}} := L^2(\widehat{X}; \mathcal{F})$$

w.r.t. the Lebesgue measure of  $\widehat{X}$ , and we denote its canonical scalar product by  $\langle \cdot, \cdot \rangle_{GF}$ . We define the isometry

$$\begin{aligned} \mathcal{H} &\longrightarrow \widehat{\mathcal{H}} \\ \Psi &\longmapsto \widehat{\Psi} := \sqrt{2\pi\rho} \Psi|_{\widehat{X}} . \end{aligned} \tag{6}$$

By  $M = -\frac{i}{2}\psi s^2\psi$  we denote the spin part of  $J$ . From  $\widehat{\Psi}$  we may recover  $\Psi$  through

$$\Psi(q, x) = \frac{1}{\sqrt{2\pi\rho}} e^{-i\alpha M} \widehat{\Psi}(0, \rho, q_3 \cos \alpha - q_4 \sin \alpha, q_4 \cos \alpha + q_3 \sin \alpha, x) . \tag{7}$$

Under the isometry (6), the corresponding transformation for the operators  $A \in \mathcal{L}(\mathcal{H})$ , i.e.,  $A \rightarrow \widehat{A} \in \mathcal{L}(\widehat{\mathcal{H}})$ , is characterized by

$$\widehat{A}\widehat{\Psi} = \widehat{A\Psi} .$$

For  $f \in C_0^\infty(X)$ , one has

$$\left( \frac{\partial}{\partial q_1} f \right) \Big|_{\widehat{X}} = \left( -\frac{i}{q_2} W_{12} f \right) \Big|_{\widehat{X}} , \tag{8}$$

where the function  $f$  is restricted to  $\widehat{X}$  only after the derivatives are performed. Applying this result to the function  $\partial f / \partial q_1$ , using the commutation relation  $[W_{12}, \partial / \partial q_1] = i\partial / \partial q_2$  and again (8), one finds

$$\left( \frac{\partial}{\partial q_1} \frac{\partial}{\partial q_1} f \right) \Big|_{\widehat{X}} = \left( \frac{1}{q_2} \frac{\partial}{\partial q_2} - \frac{1}{q_2^2} W_{12}^2 \right) f \Big|_{\widehat{X}} .$$

We set  $L := J - W_{12}$ . Then for  $\Psi \in \mathcal{H}$ ,

$$W_{12}\Psi = -L\Psi .$$

Note that  $\widehat{L} = v_3(-i\partial / \partial v_4) - v_4(-i\partial / \partial v_3) - \frac{i}{2}\psi s^2\psi$ . For  $\Psi \in \mathcal{H} \cap C_0^\infty(X; \mathcal{F})$ , a straightforward calculation yields

$$\left( -\frac{\partial}{\partial q_1} \frac{\partial}{\partial q_1} \Psi \right) = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \widehat{\Psi} + \frac{1}{2\rho^2} \widehat{\Psi} + \frac{1}{\rho^2} \widehat{L}^2 \widehat{\Psi}$$

and

$$\left( -\frac{\partial}{\partial q_2} \frac{\partial}{\partial q_2} \Psi \right) = -\frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \widehat{\Psi} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \widehat{\Psi} - \frac{3}{4} \frac{1}{\rho^2} \widehat{\Psi} .$$

As a result

$$(-\Delta \Psi)^\wedge = \left( -\Delta_{\widehat{X}} + \rho^{-2} \left( \widehat{L}^2 - \frac{1}{4} \right) \right) \widehat{\Psi} , \quad (9)$$

where  $\Delta_{\widehat{X}}$  is the formal Laplacian on  $\widehat{X}$ . We will use eq. (9) only for functions in  $C_0^\infty(\widehat{X}; \mathcal{F})$ .

We use the following partition of unity. We define

$$j_{1,t}(\xi) = \chi_r(t^{2/5}((q_1^2 + q_2^2)^{1/2} - \sqrt{2})) \cdot \chi_a(t^{2/5}(q_3, q_4, x)) , \quad \xi = (q, x) ,$$

where for  $\alpha = r, a$ , we have chosen rotation invariant functions  $\chi_\alpha \in C_0^\infty(\mathbb{R}^{n_\alpha})$  with  $n_r = 1$ ,  $n_a = 7$ ,  $0 \leq \chi_\alpha \leq 1$ ,  $\chi_\alpha(x) = 1$  if  $|x| \leq 1$  and 0 if  $|x| \geq 2$ . Let  $R \geq 1$  be fixed as  $t \rightarrow \infty$ . We choose  $j_2 \in C^\infty(X)$  with  $j_2(\xi) = j_2(|\xi|)$ ,  $0 \leq j_2 \leq 1$ ,  $j_2(\xi) = 1$  for  $|\xi| \geq 2R$  and  $j_2(\xi) = 0$  for  $|\xi| < R$ . Furthermore we set

$$j_{0,t} := (1 - j_{1,t}^2 - j_2^2)^{1/2} .$$

For technical matters we consider the embedding  $\widehat{X} \hookrightarrow \widetilde{X} := \{0\} \times \mathbb{R}^8$  and the coordinates  $(0, \eta^2, \dots, \eta^9) \in \widetilde{X}$ . By  $\eta_0$  we denote the intersection of  $\widehat{X}$  with  $\Gamma$ , i.e.,  $\eta_0 = (0, \sqrt{2}, 0, \dots, 0)$ . We define

$$\widetilde{V}_1^0(\eta) = \frac{1}{2} \sum_{\alpha, \beta=2}^9 (\text{Hess} V_1)_{\alpha\beta}(\eta_0)(\eta^\alpha - \eta_0^\alpha)(\eta^\beta - \eta_0^\beta) .$$

and introduce the following operator on  $L^2(\widetilde{X}; \mathcal{F})$

$$G_t = -\Delta_{\widetilde{X}} + t^2 \widetilde{V}_1^0 + t H_F(\eta_0) ,$$

where  $H_F(\eta_0) = -i2\sqrt{2}(\lambda_1\psi_1 + \dots + \lambda_8\psi_8) : \mathcal{F} \rightarrow \mathcal{F}$  denotes the evaluation of  $H_F$  at  $\eta_0$  and  $-\Delta_{\widetilde{X}}$  the eight dimensional Laplacian on  $\widetilde{X}$ . For  $\chi \in L^2(\widetilde{X}; \mathcal{F})$ , we define the unitary transformation

$$(T(t)\chi)(\eta) = t^2 \chi(t^{1/2}(\eta - \eta_0)) .$$

Then

$$\frac{1}{t} T(t)^* G_t T(t) = -\Delta_{\widetilde{X}} + \frac{1}{2} (\text{Hess} V_1)_{\alpha\beta}(\eta_0) \eta^\alpha \eta^\beta + H_F(\eta_0) .$$

The eigenvalue problem for this operator can be solved easily. It has purely discrete spectrum and its ground state  $\Phi^0$  has zero energy and is non degenerate: the sum of the first two terms is a harmonic oscillator, which acts on  $L^2(\widetilde{X})$  and has ground state energy  $8\sqrt{2}$ , and  $H_F(\eta_0)$  acts on  $\mathcal{F}$  and has a unique ground state with energy  $-8\sqrt{2}$ , see Appendix B (i).

Define

$$\widehat{\Psi}_t := \widehat{j}_{1,t} T(t) \Phi^0 \in C_0^\infty(\widehat{X}; \mathcal{F}) \hookrightarrow L^2(\widetilde{X}; \mathcal{F}) .$$

We recall that the corresponding  $U(1)$ -invariant wave function in  $\Psi_t \in \mathcal{H}$  is obtained using (7). Calculating the energy of this state, we find

$$\begin{aligned}\langle \Psi_t, K_t \Psi_t \rangle &= \langle \widehat{\Psi}_t, \widehat{K}_t \widehat{\Psi}_t \rangle_{GF} \\ &= \langle \widehat{\Psi}_t, (-\Delta_{\tilde{X}} + \rho^{-2}(\widehat{L}^2 - \frac{1}{4}) + t^2 \widehat{V}_1 + t \widehat{H}_F) \widehat{\Psi}_t \rangle_{GF} \\ &= \langle \widehat{\Psi}_t, G_t \widehat{\Psi}_t \rangle_{GF} + \langle \widehat{\Psi}_t, \rho^{-2}(\widehat{L}^2 - \frac{1}{4}) \widehat{\Psi}_t \rangle_{GF} \\ &\quad + \langle \widehat{\Psi}_t, t^2(\widehat{V}_1 - \widetilde{V}_1^0) \widehat{\Psi}_t \rangle_{GF} + \langle \widehat{\Psi}_t, t(\widehat{H}_F - \widehat{H}_F(\eta^0)) \widehat{\Psi}_t \rangle_{GF}.\end{aligned}\tag{10}$$

For the first term in (10), we find for  $t \rightarrow \infty$ ,

$$\begin{aligned}\langle \widehat{\Psi}_t, G_t \widehat{\Psi}_t \rangle_{GF} &= \langle T(t)\Phi^0, \widehat{j}_{1,t} G_t \widehat{j}_{1,t} T(t)\Phi^0 \rangle_{GF} \\ &= \langle T(t)\Phi^0, (\frac{1}{2} \widehat{j}_{1,t}^2 G_t + \frac{1}{2} G_t \widehat{j}_{1,t}^2 + |\nabla_{\tilde{X}} \widehat{j}_{1,t}|^2) T(t)\Phi^0 \rangle_{GF} \\ &= O(t^{4/5}),\end{aligned}$$

where we denoted the gradient on  $\tilde{X}$  by  $\nabla_{\tilde{X}}$ , and we used that  $G_t T(t)\Phi^0 = 0$  and  $\|\nabla_{\tilde{X}} j_{1,t}\|_\infty^2 = O(t^{4/5})$ . By rotation invariance of  $\Phi^0$  in the  $v_3, v_4$  variables, the second term in (10) is an order one term. The estimate

$$|t^2 \widehat{j}_{1,t}^2 (\widehat{V}_1 - \widetilde{V}_1^0)| \leq \text{const} \cdot t^2 \widehat{j}_{1,t}^2 |\eta - \eta_0|^3 \leq \text{const} \cdot t^2 \cdot t^{-6/5}\tag{11}$$

yields  $\langle \widehat{\Psi}_t, t^2(\widehat{V}_1 - \widetilde{V}_1^0) \widehat{\Psi}_t \rangle_{GF} = O(t^{4/5})$ . And a similar estimate,

$$|t \widehat{j}_{1,t}^2 (\widehat{H}_F(\eta_0) - \widehat{H}_F)| \leq \text{const} \cdot t \widehat{j}_{1,t}^2 |\eta - \eta_0| \leq \text{const} \cdot t \cdot t^{-2/5},\tag{12}$$

gives

$$\langle \widehat{\Psi}_t, t(\widehat{H}_F - \widehat{H}_F(\eta_0)) \widehat{\Psi}_t \rangle_{GF} = O(t^{3/5}),$$

as  $t \rightarrow \infty$ . Collecting terms, we find

$$\langle \Psi_t, K_t \Psi_t \rangle = O(t^{4/5}),$$

which implies that  $\lim_{t \rightarrow \infty} E_1(t)/t = 0$ . This shows the first part of (4).

To prove the second part, i.e.,

$$\liminf_{t \rightarrow \infty} E_2(t)/t \geq r > 0,\tag{13}$$

it suffices to show that there exists an  $r > 0$  such that

$$K_t \geq (t \cdot r + o(t)) \mathbb{1} + R_t,\tag{14}$$

where  $R_t$  is a symmetric, rank one operator. To see this, suppose (14) holds. Let  $\omega_{1,t}$  and  $\omega_{2,t}$  be the eigenvectors to the eigenvalues  $E_1(t)$  and  $E_2(t)$  of  $K_t$ , respectively. There exists a  $\omega_t \in \text{Span}\{\omega_{1,t}, \omega_{2,t}\}$  in the kernel of  $R_t$ . Hence

$$E_2(t) \|\omega_t\|^2 \geq \langle \omega_t, K_t \omega_t \rangle \geq (t \cdot r + o(t)) \|\omega_t\|^2$$

which implies (13).

To show (14), we use the IMS localization formula

$$K_t = \sum_{a=0}^1 j_{a,t} K_t j_{a,t} + j_2 K_t j_2 - \sum_{a=0}^1 |\nabla j_{a,t}|^2 - |\nabla j_2|^2. \quad (15)$$

Now,  $\text{supp}(j_{0,t}) \subset \{\xi \in X | \text{dist}(\xi, \Gamma) \geq t^{-2/5}\}$ . We have  $\|\nabla j_{a,t}\|_\infty^2 = O(t^{4/5})$  for  $a = 0, 1$ , and  $\|\nabla j_2\|_\infty^2 = O(1)$ . We estimate

$$\begin{aligned} j_{0,t} K_t j_{0,t} &\geq t^2 j_{0,t} V_1 j_{0,t} + t j_{0,t} H_F j_{0,t} \\ &\geq (t^2 t^{-4/5} c_V - t c_F) j_{0,t}^2, \quad \text{for some } c_V > 0, c_F > 0 \\ &\geq t \cdot r j_{0,t}^2, \quad \text{for some } r > 0, \end{aligned} \quad (16)$$

and for  $t$  large. By fixing the gauge, we have on  $L^2(\widehat{X}; \mathcal{F})$

$$\begin{aligned} \widehat{j}_{1,t} \widehat{K}_t \widehat{j}_{1,t} &= \widehat{j}_{1,t} G_t \widehat{j}_{1,t} + \widehat{j}_{1,t} t^2 (\widehat{V}_1 - \widetilde{V}_1^0) \widehat{j}_{1,t} \\ &\quad + \widehat{j}_{1,t} t (\widehat{H}_F - \widehat{H}_F(\eta_0)) \widehat{j}_{1,t} + \widehat{j}_{1,t} \rho^{-2} (-\frac{1}{4} + \widehat{L}^2) \widehat{j}_{1,t} \\ &\geq \widehat{j}_{1,t} G_t \widehat{j}_{1,t} + O(t^{4/5}) \\ &\geq \widehat{j}_{1,t} t r (1 - |T(t)\Phi^0\rangle_{GF} \cdot {}_{GF}\langle T(t)\Phi^0|) \widehat{j}_{1,t} + O(t^{4/5}), \end{aligned}$$

for some  $r > 0$ , where we have used the positivity of  $\widehat{L}^2$ , the estimates (11,12), and the gap in the spectrum of  $G_t$ . On  $\mathcal{H}$ , this yields

$$j_{1,t} K_t j_{1,t} \geq t \cdot r j_{1,t}^2 - t \cdot r |\Psi_t\rangle \cdot \langle \Psi_t| + O(t^{4/5}). \quad (17)$$

To estimate the term  $j_2 K_t j_2$ , we recall the explicit form of  $K_t$ :

$$\begin{aligned} K_t &= p_i p_i + p^\mu p^\mu + t^2 \left( |x|^2 |q|^2 + \frac{1}{4} |q|^4 + 1 - (q_1^2 + q_2^2) + (q_3^2 + q_4^2) \right) \\ &\quad + t(-ix^\mu \psi \gamma^\mu s^2 \psi + 2iq_j \lambda s^j s^2 \psi). \end{aligned}$$

We recall the notation  $\xi = (q, x)$ . Define a function  $\theta \in C^\infty(X)$  with  $\theta(\xi) = \theta(|q|)$ ,  $0 \leq \theta \leq 1$ ,  $\theta(|q|) = 1$  if  $|q| > 4$  and  $\theta(|q|) = 0$  if  $|q| < 3$ . Define  $\bar{\theta} := \sqrt{1 - \theta^2}$ . Then

$$j_2 K_t j_2 = j_2 \theta K_t \theta j_2 + j_2 \bar{\theta} K_t \bar{\theta} j_2 - j_2 (|\nabla \theta|^2 + |\nabla \bar{\theta}|^2) j_2.$$

The localization error gives order 1 contributions, i.e.,  $\|\nabla \theta\|_\infty^2 = O(1)$ ,  $\|\nabla \bar{\theta}\|_\infty^2 = O(1)$ . First we consider the case where  $|q|$  is large and estimate (see Appendix B (i) for the terms containing fermions)

$$\begin{aligned} p^\mu p^\mu &\geq 0, \quad -ix^\mu \psi \gamma^\mu s^2 \psi \geq -4|x|, \quad 2iq_j \lambda s^j s^2 \psi \geq -8|q|, \\ p_i p_i + t^2 |x|^2 |q|^2 + t(-ix^\mu \psi \gamma^\mu s^2 \psi) &\geq p_i p_i + t^2 |x|^2 |q|^2 - 4t|x| \geq 0, \end{aligned}$$

where the last inequality follows from the ground state energy of the harmonic oscillator. This yields

$$\begin{aligned} j_2 \theta K_t \theta j_2 &\geq j_2 \theta (t^2 (\frac{1}{4} |q|^4 - |q|^2 + 1) - 8t|q|) \theta j_2 \\ &\geq t^2 \cdot c j_2^2 \theta^2, \end{aligned}$$

for some  $c > 0$  and  $t$  sufficiently large. For points  $\xi = (q, x) \in \text{supp } j_2$ , if  $|q| < 4$ , then  $|x|$  is large for sufficiently large  $R$ . We have

$$\begin{aligned} j_2 \bar{\theta} K_t \bar{\theta} j_2 &\geq j_2 \bar{\theta} (p_i p_i + t^2 |x|^2 (1 - |x|^{-2}) |q|^2 - 4t|x| + t^2 - 8t|q|) j_2 \bar{\theta} \\ &\geq j_2 \bar{\theta} (t(4|x|(1 - |x|^{-2})^{1/2} - 4|x|) + t^2 - 32t) j_2 \bar{\theta} \\ &\geq t^2 \cdot c j_2^2 \bar{\theta}^2 , \end{aligned}$$

for some  $c > 0$  and  $t$  sufficiently large. Hence there exists an  $r > 0$  such that for large  $t$ ,

$$j_2 K_t j_2 \geq t \cdot r j_2^2 . \quad (18)$$

Now, inserting eqns. (16–18) into (15) yields (14) and therefore (13).  $\square$

### 4.3 Proof of Proposition 4

We decompose the Hilbert space  $\mathcal{H}_0$  as a constant fiber direct integral [17], with fiber  $F := L^2(\mathbb{R}^4; \mathcal{F})$ ,

$$\mathcal{H}_0 = \int_{\mathbb{R}^5}^{\oplus} F dx ,$$

the isomorphism being  $\Psi \mapsto (x \mapsto \Psi_x := \Psi(\cdot, x))$ . The Hamiltonian has a direct integral decomposition,

$$H_k = p^\mu p^\mu + \int_{\mathbb{R}^5}^{\oplus} H_{k,x} dx ,$$

where the fibers  $H_{k,x}$ , acting on  $F$ , are given by

$$H_{k,x} = H_x^0 + \frac{1}{4} |q|^4 + 2iq_j \lambda s^j s^2 \psi + k^2 - k(q_1^2 + q_2^2) + k(q_3^2 + q_4^2) ,$$

with

$$H_x^0 := p_i p_i + |x|^2 |q|^2 - ix^\mu \psi \gamma^\mu s^2 \psi .$$

The scalar product, the norm and operator norm in  $F$  will be denoted by  $(\cdot, \cdot)_F$  and  $\|\cdot\|_F$ , respectively. Let  $P_x$  be the projection onto the eigenspace of  $H_x^0$  corresponding to its lowest eigenvalue, which is, in fact, zero. We set  $P_x^\perp := 1 - P_x$ , and we define the projection

$$P = \int_{\mathbb{R}^5}^{\oplus} P_x dx , \quad (19)$$

and its complement  $P^\perp = 1 - P$ . As is shown in Appendix B (ii), for  $x \neq 0$ ,

$$\text{Ran } P_x = \{ \Xi_x \cdot \xi \mid \xi \in \mathcal{F} \text{ with } (\bar{u}\psi)\xi = 0, \forall u : -i\gamma^\mu x^\mu s^2 u = |x|u \} ,$$

where

$$\Xi_x(q) := (|x|\pi)^{-1} \exp\left(-\frac{1}{2}|x||q|^2\right) .$$

**Lemma 6.** *There exists an  $R > 0$  and a constant  $c > 0$  depending on  $k$ , such that for  $|x| > R$*

$$H_{k,x} \geq k^2 - c|x|^{-2}.$$

*Proof.* Since all elements in  $\text{Ran}P_x$  are spherically symmetric in  $q$  it immediately follows that

$$P_x H_{k,x} P_x \geq k^2 P_x. \quad (20)$$

We estimate, c.p. Appendix B (i),

$$2iq_j \lambda s^j s^2 \psi \geq -8|q|, \quad \text{and} \quad -k(q_1^2 + q_2^2) \geq -|x|^{-1}k(|x|^{-1}p_i p_i + |x||q|^2).$$

Hence

$$\begin{aligned} H_{k,x} &\geq |x|(1 - |x|^{-1})(|x|^{-1}p_i p_i + |x||q|^2) + |x|^{-1}p_i p_i + |x||q|^2 \\ &\quad - ix^\mu \psi \gamma^\mu s^2 \psi - 8|q| - |x|^{-1}k(|x|^{-1}p_i p_i + |x||q|^2) + k^2 \\ &\geq |x|(1 - |x|^{-1} - k|x|^{-2})(|x|^{-1}p_i p_i + |x||q|^2) - ix^\mu \psi \gamma^\mu s^2 \psi - 16|x|^{-1} + k^2, \end{aligned}$$

where we used  $|x||q|^2 - 8|q| \geq -16|x|^{-1}$  in the last inequality. The range of  $P_x^\perp$  is given by the closure of the set of linear combinations of states which are a product of an eigenstate of  $p_i p_i + |x|^2|q|^2$  and an eigenstate of  $-ix^\mu \psi \gamma^\mu s^2 \psi$ , excluding states which are a product of two ground states. Thus

$$P_x^\perp H_{k,x} P_x^\perp \geq (c^0|x| + k^2)P_x^\perp, \quad (21)$$

for some  $c^0 > 0$  and large  $|x|$ .

Using that  $\|q|^\alpha P_x\|_F \leq c_\alpha|x|^{-\alpha/2}$  from the Gaussian decay of states in  $\text{Ran}P_x$ ,

$$\begin{aligned} \|P_x^\perp H_{k,x} P_x^\perp\|_F &= \|P_x^\perp (|q|^4/4 - k(q_1^2 + q_2^2) + k(q_3^2 + q_4^2) + 2iq_j \lambda s^j s^2 \psi) P_x^\perp\|_F \\ &\leq c|x|^{-1/2} \end{aligned}$$

for some  $c > 0$  and large  $|x|$ . By the self adjointness of  $H_{k,x}$  also

$$\|P_x H_{k,x} P_x^\perp\|_F \leq c|x|^{-1/2}. \quad (22)$$

Let  $u_x \in \mathcal{D}(H_{k,x}) \subset F$ . Then from (20), (21) and (22)

$$\begin{aligned} (u_x, H_{k,x} u_x)_F &\geq k^2 \|u_x\|_F^2 + \begin{pmatrix} \|P_x u_x\|_F \\ \|P_x^\perp u_x\|_F \end{pmatrix} A_{k,x} \begin{pmatrix} \|P_x u_x\|_F \\ \|P_x^\perp u_x\|_F \end{pmatrix} \\ &\geq (k^2 + \inf_{\|\xi\|=1}(\xi, A_{k,x} \xi)) \|u_x\|_F^2, \end{aligned}$$

with

$$A_{k,x} := \begin{pmatrix} 0 & -c|x|^{-1/2} \\ -c|x|^{-1/2} & c^0|x| \end{pmatrix}.$$

We have

$$\inf_{\|\xi\|=1}(\xi, A_{k,x} \xi) \geq -c|x|^{-2},$$

for some  $c > 0$  and large  $|x|$ . Hence the Lemmma follows.  $\square$

Let  $R \geq 1$  be as in Lemma 6, and let  $\eta : \mathbb{R}^5 \rightarrow \mathbb{R}$  be a smooth function with  $\eta(x) = \eta(|x|)$ ,  $0 \leq \eta \leq 1$ ,  $\|\nabla \eta\|_\infty \leq 1$ ,  $\eta(x) = 0$  for  $|x| \leq R$  and  $\eta(x) = 1$  for  $|x| \geq 3R$ .

The deformed supercharge

$$Q_{1,k} = Q_1 + k(\gamma^1 \lambda)_2 = \frac{1}{\sqrt{2}}(D_k + D_k^\dagger) \quad (23)$$

satisfies on  $\mathcal{H}$ ,  $2(Q_{1,k})^2 = \{D_k, D_k^\dagger\} = H_k$ . Hence for  $\Psi \in \mathcal{H}$ ,  $H_k \Psi = 0$  iff  $Q_{1,k} \Psi = 0$ .

**Lemma 7.** *Let  $\Psi \in \mathcal{H}$  with  $H_k \Psi = 0$ . Then for any  $\epsilon > 0$ ,  $|x|^{-1/2-\epsilon} e^{k|x|} \eta \Psi \in \mathcal{H}$ .*

*Proof.* It is sufficient to show the claim for arbitrarily small  $\epsilon$ . To prove the lemma, we use an Agmon estimate [12]. Let  $h : \mathbb{R}^5 \rightarrow [0, \infty)$  be a smooth function such that the set

$$K = \{x \in \mathbb{R}^5 \mid k^2 - c|x|^{-2} - |\nabla h(x)|^2 < 0\}$$

is compact. Then, as we will show,

$$\int_{\mathbb{R}^5} \eta^2 \|\Psi_x\|_F^2 (k^2 - c|x|^{-2} - |\nabla h(x)|^2) e^{2h} dx \leq M_0 \|\Psi\|^2. \quad (24)$$

where

$$M_0 := \sup_{R \leq |x| \leq 3R} \left| (1 + 2|\nabla h(x)|) e^{2h(x)} \right| < \infty.$$

Define  $h_\alpha := h(1 + \alpha h)^{-1}$ . Then, by Lemma 6,

$$\begin{aligned} (\eta e^{h_\alpha} \Psi, H_k \eta e^{h_\alpha} \Psi) &\geq \int_{\mathbb{R}^5} (\eta e^{h_\alpha} \Psi_x, H_{k,x} \eta e^{h_\alpha} \Psi_x)_F dx \\ &\geq \int_{\mathbb{R}^5} \eta^2 e^{2h_\alpha} (k^2 - c|x|^{-2}) \|\Psi_x\|_F^2 dx. \end{aligned} \quad (25)$$

We estimate

$$\begin{aligned} (\eta e^{h_\alpha} \Psi, H_k \eta e^{h_\alpha} \Psi) &= 2(Q_{1,k} \eta e^{h_\alpha} \Psi, Q_{1,k} \eta e^{h_\alpha} \Psi) \\ &= 2([Q_{1,k}, \eta e^{h_\alpha}] \Psi, [Q_{1,k}, \eta e^{h_\alpha}] \Psi) \\ &\leq (|\nabla(\eta e^{h_\alpha})|^2 \Psi, \Psi) \\ &\leq ((|\nabla \eta|^2 + 2(\nabla \eta)(\nabla h_\alpha) \eta + |\nabla h_\alpha|^2 \eta^2) e^{2h_\alpha} \Psi, \Psi). \end{aligned}$$

Inserting this into inequality (25), we obtain

$$\begin{aligned} I_\alpha &:= \int_{\mathbb{R}^5} \eta^2 e^{2h_\alpha} (k^2 - c|x|^{-2} - |\nabla h_\alpha|^2) \|\Psi_x\|_F^2 dx \\ &\leq ((|\nabla \eta|^2 + 2|\nabla \eta||\nabla h_\alpha| \eta) e^{2h_\alpha} \Psi, \Psi) \\ &\leq M_0 \|\Psi\|^2. \end{aligned}$$

Using Fatou's Lemma on the set  $K^c$  and dominated convergence on  $K$  yields

$$\left( \int_K + \int_{K^c} \right) \eta^2 \|\Psi_x\|_F^2 (k^2 - c|x|^{-2} - |\nabla h|^2) e^{2h} dx \leq \liminf_\alpha I_\alpha \leq M_0 \|\Psi\|^2$$

and hence (24).

We choose  $h$  such that on the support of  $\eta$ ,  $h(x) = k|x| - \epsilon \log |x|$ . Then

$$k^2 - c|x|^{-2} - |\nabla h(x)|^2 = 2k\epsilon|x|^{-1} - (c + \epsilon^2)|x|^{-2} \geq k\epsilon|x|^{-1},$$

for large  $|x|$ . Hence, by (24)

$$\int_{\mathbb{R}^5} dx \eta^2 \|\Psi_x\|_F^2 e^{2k|x|} k\epsilon|x|^{-2\epsilon-1} < \infty,$$

which proves the Lemma.  $\square$

*Proof of Proposition 4.* We recall that the deformed Hamiltonian commutes with the action of  $Spin(5)$ . Let  $k$  be sufficiently large such that  $\Psi$  is the unique zero energy state of  $H_k$ . Thus  $\Psi$  belongs to a one dimensional representation of  $Spin(5)$ , and therefore it is  $Spin(5)$  invariant. Let  $R(S)$  denote the image of  $S$  under the canonical projection  $Spin(5) \rightarrow SO(5)$ . By  $\mathcal{R}(S)$  we denote the spin part of the  $Spin(5)$  action, i.e., the representation generated by  $-\frac{i}{4}\gamma_{ab}^{\mu\nu}(\lambda_a\lambda_b + \psi_a\psi_b)$ . Then

$$\Psi(q, R(S)x) = \mathcal{R}(S)\Psi(q, x), \quad \forall S \in Spin(5).$$

This implies that for  $x, x' \in \mathbb{R}^5$  with  $|x| = |x'|$ ,

$$|\Psi_x|_F = |\Psi_{x'}|_F.$$

We set  $\omega = x/|x|$ . Let  $d\Omega$  denote the surface measure of the unit sphere. Then

$$\begin{aligned} \int_{S^4} e^{-2k|x|\pm 2kx^1} d\Omega(\omega) &= \text{vol}(S^3) \int_0^\pi e^{-2k|x|(1\mp \cos\theta)} \sin^3 \theta d\theta \\ &\leq \text{vol}(S^3) \int_{-1}^1 e^{-2k|x|(1-\cos\theta)} 2(1-\cos\theta) d\cos\theta \\ &\leq \text{const } |x|^{-2}. \end{aligned}$$

For the ground state  $\Psi \in \mathcal{H}$  of  $H_k$ , we have

$$\begin{aligned} \int e^{\pm 2kx^1} |\Psi(q, x)|^2 dq dx &= \int e^{\pm 2kx^1} |\Psi_x|_F^2 dx \\ &= \int (1-\eta) e^{\pm 2kx^1} |\Psi_x|_F^2 dx + \int \eta e^{\pm 2kx^1} |\Psi_x|_F^2 dx \\ &\leq \text{const} + \int \eta e^{\pm 2kx^1} e^{-2k|x|} e^{2k|x|} |\Psi_x|_F^2 dx \\ &\leq \text{const} + \text{const} \int \eta |x|^{-2} e^{2k|x|} |\Psi_x|_F^2 dx \\ &< \infty, \end{aligned}$$

where in the last step we have used Lemma 7.  $\square$

## Appendix A

In this appendix we mainly follow [14]. We consider the quaternions with generators  $1, I, J, K$  satisfying the relations

$$I^2 = -1, \quad J^2 = -1, \quad K^2 = -1, \quad IJK = -1.$$

A quaternion can be expanded as

$$q = q^1 1 + q^2 I + q^3 J + q^4 K.$$

The conjugate is given by

$$\bar{q} = q^1 1 - q^2 I - q^3 J - q^4 K.$$

We note that  $q\bar{q} = \bar{q}q = |q|^2$ . By  $1^R, I^R, J^R, K^R$  we denote the matrix representation, with respect to the basis  $(1, I, J, K)$ , of the right multiplication with  $1, I, J, K$ , respectively. We have

$$I^R = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J^R = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad K^R = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that  $(AB)^R = B^R A^R$  with  $A, B \in \{1, I, J, K\}$ . We define the matrices

$$s^1 = \begin{pmatrix} 1^R & 0 \\ 0 & 1^R \end{pmatrix}, \quad s^2 = \begin{pmatrix} I^R & 0 \\ 0 & I^R \end{pmatrix}, \quad s^3 = \begin{pmatrix} J^R & 0 \\ 0 & J^R \end{pmatrix}, \quad s^4 = \begin{pmatrix} K^R & 0 \\ 0 & K^R \end{pmatrix}.$$

We remark that  $(s^l)^T = -s^l$  for  $l = 2, 3, 4$ . We choose the gamma matrices as

$$\gamma^1 = \begin{pmatrix} \mathbb{1}_{4 \times 4} & 0 \\ 0 & -\mathbb{1}_{4 \times 4} \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \mathbb{1}_{4 \times 4} \\ \mathbb{1}_{4 \times 4} & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & K^L \\ -K^L & 0 \end{pmatrix},$$

$$\gamma^4 = \begin{pmatrix} 0 & I^L \\ -I^L & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & J^L \\ -J^L & 0 \end{pmatrix},$$

with

$$I^L = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \quad J^L = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad K^L = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix},$$

where  $\sigma^i$ ,  $i = 1, 2, 3$ , are the Pauli matrices and the superscript  $L$  indicates that the matrix corresponds to left multiplication. Using that left multiplication commutes with right multiplication one sees that  $[\gamma^\mu, s^j] = 0$ .

## Appendix B

**(i)** Consider a real antisymmetric  $16 \times 16$  matrix  $S$  and the Clifford generators denoted as  $(\vartheta_1, \dots, \vartheta_8, \vartheta_9, \dots, \vartheta_{16}) = (\psi_1, \dots, \psi_8, \lambda_1, \dots, \lambda_8)$ . We will show that the map

$$i \sum_{a,b=1}^{16} \vartheta_a S_{ab} \vartheta_b : \mathcal{F} \rightarrow \mathcal{F}$$

has a ground state  $\xi \in \mathcal{F}$ , which is determined by the condition that

$$\sum_{a=1}^{16} \bar{v}_a \vartheta_a \xi = 0 \quad (26)$$

for all eigenvectors  $v$  of  $iS$  with strictly positive eigenvalue. The ground state energy is  $-\frac{1}{2}\text{tr}\sqrt{S^t S}$ . If  $S$  is invertible the ground state is unique.

The matrix  $iS$  is hermitian. Let  $v$  be an eigenvector of  $iS$  with eigenvalue  $\lambda$ , then  $\bar{v}$  is an eigenvector with eigenvalue  $-\lambda$ . Hence we have the spectral decomposition

$$iS = \sum_{j=1}^8 \lambda_j (P^{j+} - P^{j-}) , \quad \lambda_j \geq 0 ,$$

where  $P^{j\pm}$  are orthogonal projectors with  $(P^{j\pm})^t = P^{j\mp}$ . This yields

$$\begin{aligned} \sum_{a,b=1}^{16} \vartheta_a i S_{ab} \vartheta_b &= \sum_{j=1}^8 \lambda_j \sum_{a,b=1}^{16} (\vartheta_a P_{ab}^{j+} \vartheta_b - \vartheta_a P_{ab}^{j-} \vartheta_b) \\ &= \sum_{j=1}^8 2\lambda_j \sum_{a,b=1}^{16} \vartheta_a P_{ab}^{j+} \vartheta_b - \sum_{j=1}^8 \lambda_j . \end{aligned}$$

Therefore, the ground state  $\xi$  satisfies (26) and has energy  $-\sum_{j=1}^8 \lambda_j = -\frac{1}{2}\text{tr}\sqrt{S^t S}$ . If  $S$  is invertible then there are exactly 8 linearly independent eigenvectors with strictly positive eigenvalue. By the irreducibility of  $\mathcal{F}$ , the condition (26) then determines the ground state uniquely.

**(ii)** Now, let us consider the special case  $-i\psi x^\mu \gamma^\mu s^2 \psi$ . The vector  $\xi \in \mathcal{F}$  is a ground state of  $-i\psi x^\mu \gamma^\mu s^2 \psi$  if and only if

$$(\bar{u}\psi)\xi = 0$$

for all  $u$  satisfying  $-i\gamma^\mu x^\mu s^2 u = |x|u$ . We define

$$\mathcal{W}_x := \{ \xi \in \mathcal{F} \mid \xi \text{ is a ground state of } -i\psi x^\mu \gamma^\mu s^2 \psi \} .$$

The operators  $\lambda_a$  leave this space invariant and act irreducibly on it. Thus  $\dim \mathcal{W}_x = 2^4$ . The ground state of the harmonic oscillator  $p_i p_i + |x|^2 |q|^2$  is

$$\Xi_x(q) = (|x|\pi)^{-1} \exp(-\frac{1}{2}|x|q^2) .$$

By  $P_x$  we denote the projection onto the ground state of

$$H_x^0 = p_i p_i + |x|^2 |q|^2 - i \psi x^\mu \gamma^\mu s^2 \psi .$$

The harmonic oscillator part commutes with the fermionic part. The ground state energy of  $H_x^0$  is zero and

$$\text{Ran}P_x = \{ \Xi_x \cdot \xi \mid \xi \in \mathcal{F} \text{ with } (\bar{u}\psi)\xi = 0, \forall u : -i\gamma^\mu x^\mu s^2 u = |x|u \} .$$

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